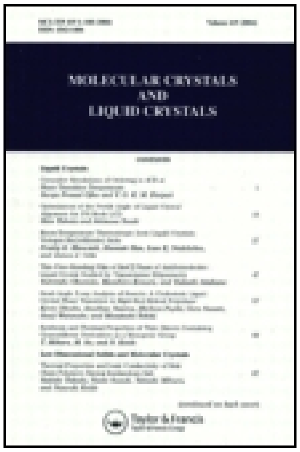

Molecular Crystals and Liquid Crystals Science and Technology. Section A. Molecular Crystals and Liquid Crystals

Publication details, including instructions for authors and subscription information:
http://www.tandfonline.com/loi/gmcl19

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Version of record first published: 04 Oct 2006

To cite this article: Peng-Ye Wang (1998): First-Passage-Time Distributions of the Freedericksz Transition in Nematic Liquid Crystals, Molecular Crystals and Liquid Crystals Science and Technology. Section A. Molecular Crystals and Liquid Crystals, 319:1, 1-17

To link to this article: http://dx.doi.org/10.1080/10587259808045643

Mol. Cryst. Liq. Cryst., Vol. 319, pp. 1–17
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First-Passage-Time Distributions of the Freedericksz Transition in Nematic Liquid Crystals

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(Received 31 January 1997; In final form 18 September 1997)

A two-stage transient statistical investigation of the Freedericksz transition in Nematic Liquid Crystals (NLC) is presented. After applying an external field that exceeds the Freedericksz critical value, the director of the NLC relaxes from its unstable equilibrium state with the effect of noise. The statistical properties of this transient process can be characterized by the First Passage Time (FPT). We separate the whole process into two stages with the time when the distribution function of the director has its variance equal to the variance of the steady-state distribution at the critical point. The first stage is considered as an Ornstein-Uhlenbeck process and the second stage as a deterministic nonlinear transformation of the distribution function of the director. The explicit expression of the FPT distribution is analytically derived with this consideration. Monte Carlo simulations shown that the present theory describes the transient behavior of the NLC much better than the previous asymptotic approach. The mean, the variance and the skewness of our explicit FPT distribution function show excellent agreement with the numerical simulation.

Keywords: Nematic liquid crystal; first passage time; Freedericksz transition; transient process

1. INTRODUCTION

Director reorientation of nematic liquid crystals (NLC) under the action of external fields (electric, magnetic or optical field) are studied quite extensively [1–3]. However, most investigations on this effect are based on the deterministic theory. In fact, the effect of stochastic fluctuations,

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e.g. thermal noise, on the director of the liquid crystal molecule is not avoidable [1 – 3]. Therefore, the dynamics of Freedericksz transition in NLC is a typical decay process of an unstable equilibrium state affected by noise. The transient behavior of similar unstable equilibrium states has been investigated in various systems, such as lasers [4 – 8], super-radiance [9, 10], chemical reaction [11, 12], spinodal decomposition [13, 14], etc. First-Passage-Time (FPT) statistics is a powerful method to investigate these non-equilibrium systems. In the case of a random magnetic field, the transient dynamics of the Freedericksz transition were studied theoretically some time ago by Sagúes *et al.* [15] where an asymptotic approximation to the FPT distribution was given following the procedure used by Haake *et al.* [16].

Since the classic FPT problem [17] has so far proved solvable only for the simplest random process, a good approach is important in deriving the distribution function. In our present statistical analysis of the Freedericksz transition a two-stage consideration is used. We divide the transient process into two stages by comparing the variance of the time-dependent distribution of the nematic director to that of the steady-state distribution at the Freedericksz critical point. Due to the fact that if the initial state is close to or above the critical point the noise in the path of the process is negligible, the first stage is considered as an Ornstein-Uhlenbeck process while the second as a nonlinear deterministic transformation of the distribution function obtained in the first stage. Under the two-stage considerations the explicit expression of the FPT distribution function is obtained analytically. In the first stage, because the deformation of the nematic director is very small and therefore, far away from saturation, the process is well described by a linear Langevin equation of which the exact non-stationary (or time-dependent) distribution function can be obtained by analytically solving the corresponding Fokker-Planck equation [18]. When the variance of the obtained distribution function, which is increasing with time, reaches the value of the variance of the steady-state distribution function at the critical point, the first stage finishes and the second stage starts. In the second stage because the initial distribution is as wide as the critical one and the effect of the random force in the decay process is sufficiently weak compared with the effect of the initial state, the transition of the system from its initial state can be described by using a deterministic transformation of the random initial conditions. Although the analysis is for the Freedericksz transition, the result turns out to be rather universal for similar stochastic systems with a cubic saturation. Comparing the present result and the asymptotic one [15] with Monte Carlo simulations shows that the present theory is more appropriate in characterizing the NLC system.

The calculation of the mean, the variance and the skewness of the FPT with the present distribution function gives results in excellent agreement with the Monte Carlo data.

The paper is organized as follows. The Langevin equation is given in Sec. II. The corresponding Fokker-Planck equation and its stationary solution are obtained. In Sec. III the FPT distribution function is approximated with a deterministic transformation of the random initial conditions, where the noise in the path is completely omitted. In Sec. IV we introduce the two-stage consideration and derive the FPT distribution function. In Sec. V some Monte Carlo results are given to verify the theory. For comparison, some data of the previous asymptotic theory are also given. Finally, Sec. VI contains the conclusions.

2. NONLINEAR LANGEVIN EQUATION AND STATIONARY DISTRIBUTION FUNCTION

With the deterministic considerations, the dynamics of the Freedericksz transition of the NLC can be described with the equation of motion of the director [1]:

$$\gamma \frac{\partial \theta}{\partial t} = k \frac{\partial^2 \theta}{\partial z^2} + \chi_a H^2 \sin\theta \cos\theta, \tag{1}$$

where k is the elastic constant, χ_a is the anisotropy of the susceptibility, γ is the coefficient of the viscosity, H is the magnetic field.

Equation (1) is exactly valid for twist deformation and also valid for splay and bend deformations in the one-constant approximation [1]. The expression for magnetic field is given here. If one makes the substitution

$$\frac{\varepsilon_a E^2}{8\pi} \rightarrow \frac{1}{2} \chi_a H^2$$

in Eq. (1) the expression for electric field can be easily obtained.

Following the usual approximation [1–3], we assume that the maximum external field does not greatly exceed the critical field

$$H_c = \frac{\pi}{d} \sqrt{\frac{k}{\chi_a}}, \tag{2}$$

where d is the thickness of the NLC sample. In this case the angle of deformation of the director is small and $\sin\theta \cos\theta$ can be expanded in powers of θ. We have, therefore

$$\gamma \frac{\partial \theta}{\partial t} = k \frac{\partial^2 \theta}{\partial z^2} + \chi_a H^2 \left(\theta - \frac{2}{3} \theta^3 \right). \tag{3}$$

With a rigid boundary condition, i.e. $\theta = 0$ at the surface of the NLC, to a good approximation, the solution of Eq. (3) can be taken as

$$\theta = \theta_m(t) \cos \frac{\pi z}{d}. \tag{4}$$

Substitution of Eq. (4) in Eq. (3) and multiplying Eq. (3) by $\cos(\pi z/d)$ and performing the integration over z, one obtains the equation for $\theta_m(t)$

$$\gamma \frac{d\theta_m}{dt} = \left(\frac{H^2}{H_c^2} - 1 \right) \theta_m - \frac{1}{2} \frac{H^2}{H_c^2} \theta_m^3, \tag{5}$$

or, dropping the subscript m of $\theta_m(t)$ for clarity

$$\frac{d\theta}{dt} = (h^2 - 1)\theta - \frac{1}{2} h^2 \theta^3, \tag{6}$$

or, for simplicity

$$\frac{d\theta}{dt} = A\theta - B\theta^3, \tag{7}$$

where

$$h^2 = \frac{H^2}{H_c^2}, \quad A = h^2 - 1, \quad B = \frac{1}{2} h^2 \tag{8}$$

and the time t is normalized to γ^{-1}.

Because the NLC director is subjected to complicated random forces, termed noise, we introduce a stochastic term in Eq. (7) accounting for the random fluctuation of the system. This leads to a nonlinear Langevin equation:

$$\frac{d\theta}{dt} = A\theta - B\theta^3 + af(t), \tag{9}$$

where $f(t)$ is Langevin force assumed to be a Gaussian random variable with zero mean and δ correlation function. The constant a describes the noise strength. We choose the following normalization:

$$\langle f(t)\rangle = 0; \quad \langle f(t)f(t')\rangle = 2\delta(t-t'). \tag{10}$$

The analytical solution of the stochastic differential Eq. (9) can not be obtained. We can setup a Fokker-Planck equation [18] by which the probability density $W(\theta)$ of the stochastic variable θ can be calculated. In the Fokker-Planck equation, because the Langevin force is assumed to be δ-correlated and Gaussian distributed, the Kramers-Moyal coefficients vanish for $n \geq 3$ and only the drift and diffusion coefficient enter:

$$D^{(1)} = A\theta - B\theta^3; \quad D^{(2)} = a^2. \tag{11}$$

For the stationary distribution function of θ we get the Fokker-Planck equation:

$$a^2 \frac{dW(\theta)}{d\theta} = (A\theta - B\theta^3)W(\theta). \tag{12}$$

Integrating Eq. (12) gives the distribution function:

$$W(\theta) = Ce^{\left(\frac{1}{2}A\theta^2 - \frac{1}{4}B\theta^4\right)/a^2}, \tag{13}$$

where C is an integration constant which can be determined by the normalization condition:

$$\int_{-\infty}^{+\infty} W(\theta)\, d\theta = 1. \tag{14}$$

Figure 1 shows the normalized distribution function (13) in the cases of the external field H below ($h^2 < 1$), at ($h^2 = 1$) and above ($h^2 > 1$) the critical field H_c of the Freedericksz transition, where we take $a^2 = 10^{-5}$. It can be seen that in the case of $h^2 \leq 1$ the distribution function has only one maximum at $\theta = 0$ while in the case of $h^2 > 1$ the distribution function has two maxima at $\theta = \pm\sqrt{A/B}$ which is consistent with the deterministic theory when $a^2 = 0$. We can also see that the closer the field to H_c the larger the uncertainty of the director, which means that the NLC molecules are most sensitive to noise disturbance at the Freedericksz critical point.

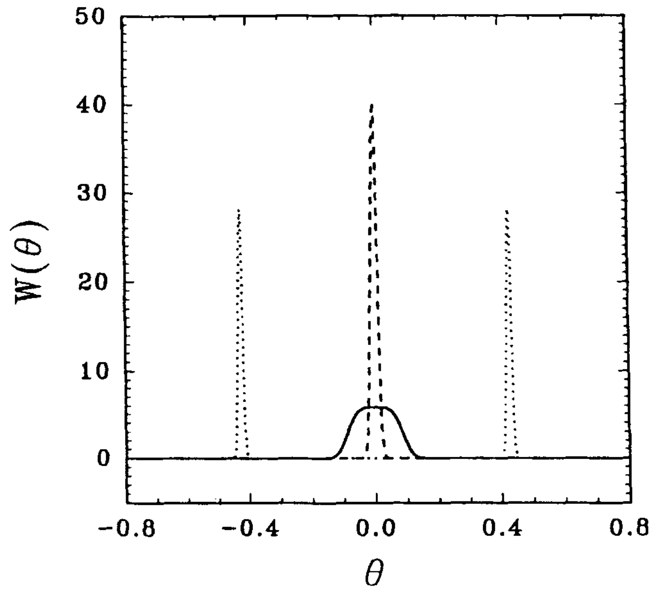

FIGURE 1 Distribution function of the angle of the NLC director in three cases: dashed curve: $h^2 = 0.9$ (below critical point); solid curve: $h^2 = 1.0$ (at critical point); dotted curve: $h^2 = 1.1$ (above critical point).

3. APPROXIMATE FIRST-PASSAGE-TIME DISTRIBUTION FUNCTION GIVEN BY DETERMINISTIC TRANSFORMATION OF RANDOM INITIAL CONDITION

Having obtained the explicit expression of the stationary distribution function $W(\theta)$ we are now in a position to evaluate the FPT distribution function for a prescribed threshold which will give a statistical description of the transient (or turn-on) time of the NLC. If a step external field (with final value of $h = H/H_c$) is applied to the NLC the director will develop from an initial angle having a distribution function $W(\theta)|_{h=h_0}$ where $h_0 = H_0/H_c$ and H_0 is the initial value of the step field (the asymptotic result of the FPT distribution in Ref. [15] is given only for $h_0 = 0$).

In this section we first neglect the effect of the noise in the transition duration (i.e. in the path). Based on this approximation, we derive the time dependence of the angle $\theta(t)$ by integrating Eq. (9) omitting the Langevin force and with a sotchastic initial value. We get

$$\theta^2(t) = \frac{\theta^2(\infty)}{1 + \left(\dfrac{\theta^2(\infty)}{\theta^2(0)} - 1\right)e^{-2At}}, \tag{15}$$

where $\theta(\infty) = \sqrt{A/B}$ is the final steady-state value of $\theta(t)$, and $\theta(0)$ is the stochastic initial steady-state angle having the distribution function $W(\theta(0))$ with the initial field h_0. From Eq. (13) we have

$$W(\theta(0)) = Ce^{\left(\frac{1}{2}A_0\theta^2(0)-\frac{1}{4}B_0\theta^4(0)\right)/a^2}, \tag{16}$$

where $A_0 = h_0^2 - 1$ and $B_0 = h_0^2/2$.

As mentioned above, the FPT in the NLC system is defined as the time interval between the moment when the step external field is applied and the moment when the angle $\theta(t)$ reaches a prescribed threshold value θ_{th} for the first time. Let the threshold angle be

$$\theta_{\text{th}} = b\theta(\infty), \tag{17}$$

where $0 < b < 1$ because $0 < \theta_{\text{th}} < \theta(\infty)$. Substitution of Eq. (17) in Eq. (15) gives

$$\theta^2(0) = \frac{A/B}{1 + \left(\frac{1}{b^2} - 1\right)e^{2At_1}}, \tag{18}$$

where t_1 is the FPT.

Since the distribution function of $\theta(0)$ is known, we can derive the distribution function of t_1 with the transformation relation:

$$W(t_1) = W(\theta(0))\frac{d\theta(0)}{dt_1}. \tag{19}$$

Combination of Eq. (16), Eq. (18) and Eq. (19) gives the distribution function of the FPT:

$$\begin{aligned} W(t_1) = 2C\sqrt{\frac{A^3}{B}}\left(\frac{1}{b^2} - 1\right)\left(1 + \left(\frac{1}{b^2} - 1\right)e^{2At_1}\right)^{-3/2} \\ \times \exp\left(2At_1 + \frac{A_0}{2a^2}\frac{A/B}{1 + \left(\frac{1}{b^2} - 1\right)e^{2At_1}}\right. \\ \left. - \frac{B_0}{4a^2}\left(\frac{A/B}{1 + \left(\frac{1}{b^2} - 1\right)e^{2At_1}}\right)^2\right). \end{aligned} \tag{20}$$

In order to see the degree of approximation of Eq. (20), in Figure 2 we plot both the analytical result of Eq. (20) and the Monte Carlo data of the numerical simulation with Eq. (9). (Details of the numerical simulation will

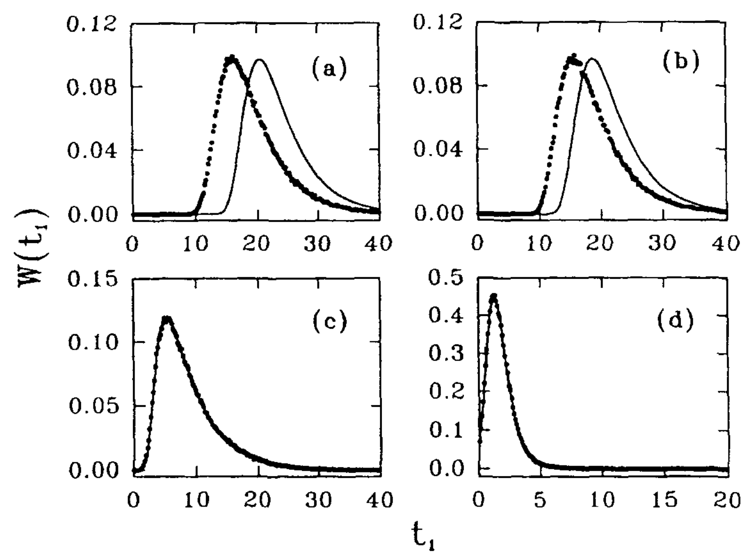

FIGURE 2 Plot of the FPT distribution function (20) (solid curves) and Monte Carlo data (dots) for $h^2 = 1.2$, $a^2 = 10^{-5}$, $b^2 = 0.1$ and (a) $h_0^2 = 0$, (b) $h_0^2 = 0.5$, (c) $h_0^2 = 1.0$ and (d) $h_0^2 = 1.01$.

be given later in Sec. V). It can be seen that when the initial field is very close to or above the critical value (i.e. $h_0 \approx 1$ or $h_0 > 1$), Eq. (20) agrees well with the Monte Carlo result, while for the case when the initial field is below the critical value (usually $h_0 = 0$) deviation is apparent.

4. TWO-STAGE CONSIDERATION FOR THE FPT DISTRIBUTION FUNCTION

From Figure 2 we can see that when the initial field is below the critical value (usual case) the effect of noise in the path of the transient process can not be neglected. The fact that the noise in the path has little effect on the system if it starts from the critical state gives a clue that when the initial field is below the critical value the whole transient process can be divided into two stages by a moment t_0 when the time-dependent distribution function of the director angle approaches the critical steady-state distribution function.

In the first stage, because the deformation of the director is small, the saturation term $B\theta^3$ in Eq. (9) can be omitted with a good approximation. Therefore, Eq. (9) becomes

$$\frac{d\theta}{dt} = A\theta + af(t), \tag{21}$$

which is a Ornstein-Uhlenbeck process with a linear drift coefficient and a constant diffusion coefficient, i.e.

$$D^{(1)} = A\theta; \quad D^{(2)} = a^2. \tag{22}$$

The equation for the transition probability $P(\theta, t|\theta', t')$ is then

$$\frac{\partial P}{\partial t} = -A\frac{\partial}{\partial \theta}(\theta P) + a^2\frac{\partial^2 P}{\partial \theta^2}, \tag{23}$$

with the initial condition

$$P(\theta, t'|\theta', t') = \delta(\theta - \theta'). \tag{24}$$

By making a Fourier transform with respect to θ the solution of Eq. (23) is obtained:

$$P(\theta, t|\theta', t') = \sqrt{\frac{A}{2\pi a^2(e^{2A(t-t')} - 1)}}\exp\left(\frac{A(\theta - \theta' e^{A(t-t')})^2}{2a^2(1 - e^{2A(t-t')})}\right). \tag{25}$$

The general solution for the probability density with the initial distribution function $W(\theta', t')$ is then given by

$$W(\theta, t) = \int P(\theta, t|\theta', t')W(\theta', t')d\theta'. \tag{26}$$

Omitting the saturation term, from Eq. (16), we have the normalized initial distribution

$$W(\theta') = \left(\frac{-A_0}{2a^2\pi}\right)^{\frac{1}{2}} e^{\frac{1}{2}A_0\theta'^2/a^2}. \tag{27}$$

By inserting this expression and Eq. (25) in Eq. (26) we can perform the integration, obtaining

$$W(\theta, t) = \sqrt{\frac{-A_0 A}{2\pi a^2((A - A_0)e^{2At} + A_0)}}\exp\left(\frac{A_0 A\theta^2}{2a^2((A - A_0)e^{2At} + A_0)}\right), \tag{28}$$

where the initial time t' is taken as zero without loss of generality. This is a Gaussian distribution function with zero mean and the time-dependent variance

$$\sigma^2(t) = a^2((A - A_0)e^{2At} + A_0)/(-A_0 A). \tag{29}$$

A three-dimensional plot of Eq. (28) is shown in Figure 3 from which we can see that the distribution function widens rapidly as a function of time.

As mentioned in Sec. I we assume that the first stage finishes at the moment when the variance $\sigma^2(t)$ reaches the variance of the steady-state distribution at the Freedericksz critical point. Let the time be t_0 at this moment. At the critical point we have $H=H_c$ and then $h_c=1$, $A_c=0$ and $B_c=1/2$ (the subscript c stands for the critical point). Substituting these values into Eq. (13) we obtain the normalized critical steady-state distribution function

$$W_c(\theta)=\frac{2}{\Gamma(1/4)}\left(\frac{1}{8a^2}\right)^{1/4}e^{-\theta^4/8a^2} \tag{30}$$

and its variance

$$V\theta|_c=\int_{-\infty}^{\infty}\theta^2W_c(\theta)d\theta=\frac{\Gamma(3/4)}{\Gamma(1/4)}\sqrt{8a^2}, \tag{31}$$

where $\Gamma(1/4)$ and $\Gamma(3/4)$ are gamma functions [19].

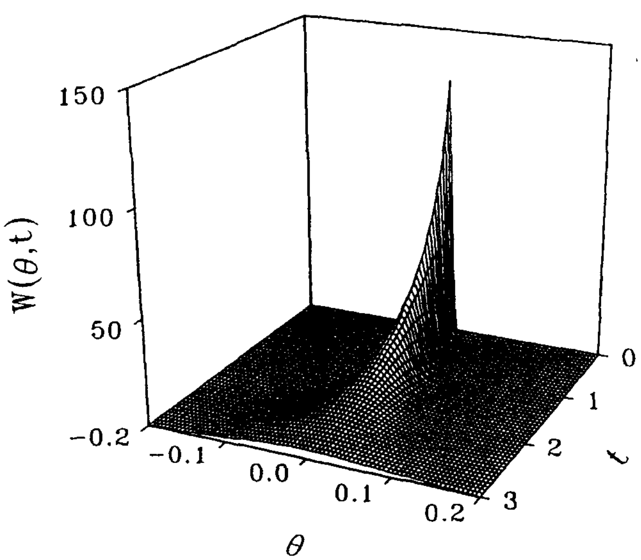

FIGURE 3 Temporal evolution of the distribution function (28) of the director where $h_0^2=0$, $h^2=2$ and $a^2=10^{-5}$.

Equalizing $\sigma^2(t)$ in Eq. (29) and $V\theta|_c$ in Eq. (31) for the time t_0, i.e.

$$\sigma^2(t_0) = V\theta|_c, \tag{32}$$

we obtain

$$t_0 = \frac{1}{2A}\ln\left(\frac{-A_0}{A-A_0}\left(1+\frac{\sqrt{8}A}{\sqrt{a^2}}\frac{\Gamma(3/4)}{\Gamma(1/4)}\right)\right). \tag{33}$$

The final distribution function of the first stage is then obtained by replacing the time t with t_0 in Eq. (28), i.e.

$$W(\theta, t_0) = \sqrt{\frac{-A_0A}{2\pi a^2((A-A_0)e^{2At_0}+A_0)}}\exp\left(\frac{A_0A\theta^2}{2a^2((A-A_0)e^{2At_0}+A_0)}\right). \tag{34}$$

Having obtained the final state of the first stage we are now in a position to consider the process of the second stage. Eq. (34) will act as the initial distribution function for $\theta(t_0)$ in the second stage because the end of the first stage is just the beginning of the second stage. Since the variance of this initial distribution function is equal to that of the critical steady-state distribution, we suppose that the initial state for the second stage is not greatly different from the steady state at the critical point. From the result of Sec. III we have seen that in the case where the system starts from the critical point, the effect of noise in the path can be neglected and the FPT distribution function can be derived using deterministic transformation of the random initial condition. Therefore, we can treat the second stage in the same way. From the deterministic relation Eq. (18), replacing $\theta(0)$ with $\theta(t_0)$, we obtain

$$\theta^2(t_0) = \frac{A/B}{1+\left(\frac{1}{b^2}-1\right)e^{2At'_1}}. \tag{35}$$

where t'_1 is the FPT for the second stage, i.e. the time interval between the time t_0 and the moment when the variable $\theta(t)$ reaches the prescribed threshold θ_{th} for the first time. The toal FPT is then the sum of the first stage transient time t_0 and the FPT of the second stage, i.e.

$$t_1 = t_0 + t'_1. \tag{36}$$

With the same transformation relation of the distribution as Eq. (19), i.e.

$$W(t'_1) = W(\theta, t_0)\frac{d\theta(t_0)}{dt'_1}, \tag{37}$$

and, from Eq. (35)

$$\frac{d\theta(t_0)}{dt'_1} = \pm\sqrt{\frac{A^3}{B}}\left(\frac{1}{b^2}-1\right)\left(1+\left(\frac{1}{b^2}-1\right)e^{2At'_1}\right)^{-3/2} e^{2At'_1}, \tag{38}$$

because $\theta(t_0) > 0$ and $\theta(t_0) < 0$ are identical for the FPT if they have the same absolute value, we then have

$$W(t'_1) = 2W(\theta, t_0)\left|\frac{d\theta(t_0)}{dt'_1}\right|. \tag{39}$$

Substitution of Eq. (34), Eq. (35) and (38) into Eq. (39) gives

$$\begin{aligned} W(t'_1) = {} & \frac{2A^2}{\sqrt{B}}\left(\frac{-A_0}{2a^2\pi((A-A_0)e^{2At_0}+A_0)}\right)^{1/2} \\ & \times\left(\frac{1}{b^2}-1\right)\left(1+\left(\frac{1}{b^2}-1\right)e^{2At'_1}\right)^{-3/2} \\ & \bullet\exp\left(2At'_1+\frac{A_0A^2}{2a^2B((A-A_0)e^{2At_0}+A_0)\left(1+\left(\frac{1}{b^2}-1\right)e^{2At'_1}\right)}\right). \end{aligned} \tag{40}$$

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With Eq. (36) we finally obtain the FPT distribution function

$$\begin{aligned} W(t_1) = {} & \frac{2A^2}{\sqrt{B}}\left(\frac{-A_0}{2a^2\pi((A-A_0)e^{2At_0}+A_0)}\right)^{1/2} \\ & \times\left(\frac{1}{b^2}-1\right)\left(1+\left(\frac{1}{b^2}-1\right)e^{2A(t_1-t_0)}\right)^{-3/2} \\ & \bullet\exp\left(2A(t_1-t_0)+\frac{A_0A^2}{2a^2B((A-A_0)e^{2At_0}+A_0)\left(1+\left(\frac{1}{b^2}-1\right)e^{2A(t_1-t_0)}\right)}\right). \end{aligned} \tag{41}$$

With this explicit expression of the distribution function, any average of the FPT can be obtained by integration easily. Detailed results will be discussed in the next section.

5. COMPARISON WITH NUMERICAL SIMULATIONS

In order to verify our analytical result, we have performed Monte Carlo simulations based on the original stochastic Eq. (9). From the Monte Carlo data it is possible to evaluate the FPT distribution function, the mean, the variance, and the skewness of the FPT, all of which can be compared with those calculated with the analytical expression (41). Some data of previous asymptotic results are also shown for comparison.

We simulate the Langevin force on a computer, integrate the equation of motion with the simulated Langevin force and then take the average for a large number of realizations as much as 2.5×10^5. The time step of the integration is taken as $\tau=0.01$. With an initial value $\theta(0)=\theta_0$ at $t=0$ and the discrete time $t_n=n\tau$ $(n=1,2,3\ldots)$ the temporal evolution of the stochastic variable can be calculated according to the iteration relation

$$\theta_{n+1} = \theta_n + D^{(1)}(\theta_n, t_n)\tau + \sqrt{D^{(2)}(\theta_n, t_n)\tau}\xi_n, \tag{42}$$

where $\theta_n=\theta(t_n)$, $D^{(1)}$ and $D^{(2)}$ are the drift and diffusion coefficient, respectively, given by Eq. (11), and ξ_n are independent Gaussian-distributed random variables with zero mean and with variance 2.

The random variables ξ_n are obtained using the relation

$$\xi_n = \sqrt{\frac{24}{M}}\sum_{i=1}^{M}\left(r_n - \frac{1}{2}\right), \tag{43}$$

where r_n are random numbers in the range of $0\leq r_n<1$ generated in the computer and M is a large number which is taken as 10 in the simulation.

In the simulation of the FPT distribution, for any given initial field we first determine the starting value of the transient process, i.e. the initial angle $\theta(0)=\theta_0$. Since this initial angle must satisfy the Langevin equation (9), we first use Eq. (42) with parameters corresponding to the initial field and with any small initial value, e.g. zero, to calculate to a long enough time for reaching the steady state. This steady-state angle which is a random variable is taken as the starting value for one realization. Then the final field which exceeds the Freedericksz critical value is applied to the NLC inducing a sudden change of the parameters A and B in Eq. (9). With the new parameter values the temporal evolution of the angle $\theta(t)$ is again calculated with Eq. (42) but with the initial angle just obtained. At the moment when $\theta(t)$ first reaches the prescribed threshold the calculation for this realization terminates. The time interval between this moment and the moment when the final field is applied is recorded as the FPT. Repeating the realization for 2.5×10^5 times the distribution can be obtained.

A typical FPT distribution function is shown in Figure 4 where the solid curve is obtained with Eq. (41). It can be seen that the analytical result agrees well with the Monte Carlo data. For comparison, the result of previous asymptotic analysis [15] is also plotted with dashed curve showing appreciable deviation with the numerical simulation.

With the FPT distribution function it is easy to calculate the mean $Mt_1\equiv\langle t_1\rangle$, the variance $Vt_1\equiv\langle(\Delta t_1)^2\rangle$, and the skewness $St_1\equiv\langle(\Delta t_1)^3\rangle$, where $\Delta t_1\equiv t_1-\langle t_1\rangle$. In order to give a more detailed comparison between our analytical result and the Monte Carlo data, we have plotted the mean, the variance, and the skewness of the FPT obtained both with the explicit

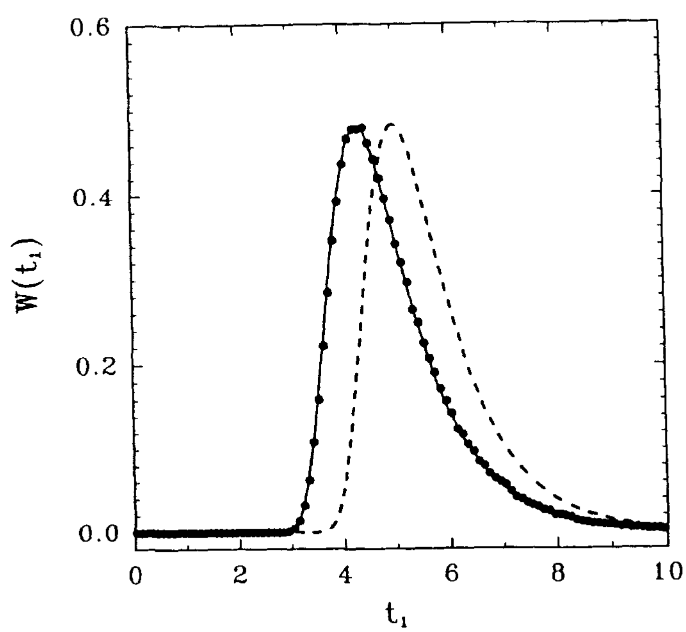

FIGURE 4 A typical FPT distribution function where $h_0^2=0$, $h^2=2$, $a^2=10^{-5}$ and $b^2=0.1$. Solid curve: analytical result of the present two-stage consideration; Dots: Monte Carlo data; Dashed curve: result of the asymptotic approximation.

distribution function (41) and the numerical simulation versus the intensity of the initial field h_0^2 in Figure 5.

From Figure 5 we can see that the mean decreases with increasing initial intensity while the variance and the skewness almost keep constant. As we know from the analyses in Sec. II when the initial field is closer to the critical value the distribution of the initial angle is wider, which means that the transient process starts with a greater possibility from a larger angle and, therefore, inducing a shorter passage time in the first stage. For a lower initial field intensity, e.g. $h_0 = 0$, the transient process spends a longer time in the development of the width of the distribution of the angle, which induces a longer passage time in the first stage. Therefore, the effect of h_0 on the FPT is only in the first stage. This is also the reason why the variance and skewness of the FPT remain unchanged. From the two-stage analysis we can

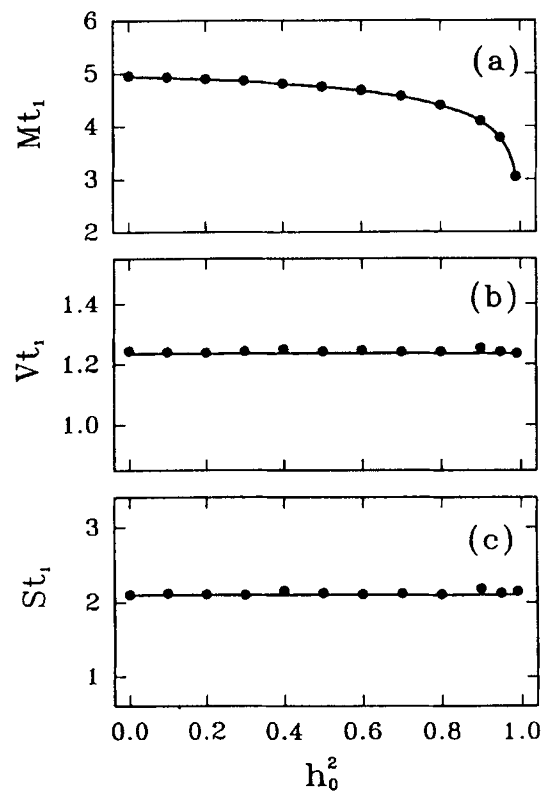

FIGURE 5 (a) The mean, (b) the variance, and (c) the skewness of the FPT versus the intensity of the initial field h_0^2, where $h^2 = 2$, $a^2 = 10^{-5}$, and $b^2 = 0.1$. Solid curves: analytical results; Dots: Monte Carlo data.

see that the first stage only gives a time duration for the Gaussian distribution function (28) to develop its variance to equal to the variance of the critical steady-state distribution. We should note that in the previous asymptotic approximation the effect of h_0 is omitted.

Similar phenomenon can be found when we change the intensity of the random Langevin force. The mean is decreased with increasing a^2. We can see that equation (29) gives the variance $\sigma^2(t)$ proportional to a^2 while in Eq. (31) the variance of the critical steady-state is proportional to $\sqrt{a^2}$, which will induce a shorter time duration in the first stage because $\sigma^2(t)$ increases faster with increasing a^2.

All calculations indicate that our explicit FPT distribution function obtained with the two-stage approximation is in good agreement with the Monte Carlo simulation, while the asymptotic approaches show appreciable deviation.

6. CONCLUDING REMARKS

In this paper, we have analyzed the FPT distribution of the Freedericksz transition in NLC with a two-stage consideration. The first stage is considered as an Ornstein-Uhlenbeck process while the second stage as a nonlinear deterministic transformation of the distribution function obtained in the first stage. The explicit distribution function of FPT has been obtained analytically with this consideration. Comparisons of the distribution function, the mean, the variance and the skewness of the FPT of the analytical result with those of the Monte Carlo simulations show that the present two-stage consideration is an excellent approximation. Although our analysis is on the NLC system, the result turns out to be universal for a stochastic system with a cubic saturation and an additive Gaussian-distributed white noise as the Langevin force.

Acknowledgment

This research was supported by the National Natural Science Foundation of China.

References

[1] P. G. de Gennes and J. Prost, *The Physics of Liquid Crystals* (Clarendon press, Oxford, 1993).
[2] L. M. Blinov, *Electro-Optical and Magneto-Optical Properties of Liquid Crystals* (John Wiley and Sons Ltd., Chichester, 1983).

[3] P. Pieranski, F. Brochard and E. Guyon, *J. Phys. (Paris)*, **33**, 681 (1972); *ibid*, **34**, 35 (1973); F. Brochard, P. Pieranski and E. Guyon, *Phys. Rev. Lett.*, **28**, 1681 (1972); B. Ya. Zeldovich, N. V. Tabiryan and Yu. S. Chilingaryan, *Sov. Phys. JETP*, **54**, 32 (1981); N. V. Tarbiryan, A. V. Sukhov and B. Ya. Zeldovich, *Mol. Cryst. Liq. Cryst.*, **136**, 1 (1986); S. D. Durbin, S. M. Arakelian and Y. R. Shen, *Phys. Rev. Lett.*, **47**, 1411 (1981); H. L. Ong, *Phys. Rev. A*, **28**, 2393 (1981); Peng-Ye Wang, Hong-Jun Zhang and Jian-Hua Dai, *Opt. Lett.*, **12**, 654 (1987); Peng-Ye Wang, Jian-Hua Dai and Hong-Jun Zhang, *Phys. Rev. A*, **41**, 3250 (1990).
[4] F. T. Arecchi and E. O. Schulz-Dubois (Eds.), *Laser Handbook* (North-Holland, Amsterdam, 1972).
[5] F. T. Arecchi, V. Degiorgio and B. Querzola, *Phys. Rev. Lett.*, **19**, 1168 (1967); F. T. Arecchi and V. Degiorgio, *Phys. Rev. A*, **3**, 1108 (1971); J. P. Gordon and E. W. Aslarksen, *IEEE J. Quantum Electron*, **6**, 428 (1970).
[6] F. T. Arecchi and A. Politi, *Phys. Rev. Lett.*, **45**, 1219 (1980); R. Roy, A. W. Yu and S. Zhu, *Phys. Rev. Lett.*, **55**, 2794 (1985); P. Spano, A. Mecozzi and A. Sapia, *Phys. Rev. Lett.*, **64**, 3003 (1990).
[7] M. Suzuki, *J. Stat. Phys.*, **16**, 477 (1977); F. Haake, *Phys. Rev. Lett.*, **41**, 1685 (1978); F. de Pasquale, J. M. Sancho, M. San Miguel and P. Tartaglia, *Phys. Rev. Lett.*, **56**, 2473 (1986).
[8] M. Ciofini, A. Lapucci, R. Meucci, Peng-Ye Wang and F. T. Arecchi, *Phys. Rev. A*, **46**, 5874 (1992); *Europhys. Lett.*, **21**, 735 (1993).
[9] D. Polder, M. F. H. Schuurmans and Q. H. F. Vrehen, *Phys. Rev. A*, **19**, 1192 (1979).
[10] F. Haake, H. King, J. Haus, G. Schröder and R. Glauber, *Phys. Rev. Lett.*, **45**, 558 (1980); *Phys. Rev. A*, **23**, 1322 (1981).
[11] I. Prigogine and G. Nicolis, *J. Chem. Phys.*, **46**, 3542 (1967); I. Prigogine and R. Lefever, *J. Chem. Phys.*, **48**, 1695 (1968); R. Lefever, *J. Chem. Phys.*, **49**, 4977 (1968).
[12] Y. Kuramoto and T. Tsuzuki, *Prog. Theor. Phys.*, **52**, 1399 (1974); **54**, 60 (1975); H. Mashiyama, A. Ito and T. Ohta, *Prog. Theor. Phys.*, **54**, 1050 (1975).
[13] J. S. Langer, M. Bar-on and H. D. Miller, *Phys. Rev. A*, **11**, 1417 (1975).
[14] K. Kawasaki, M. C. Yalabik and J. D. Gunton, *Phys. Rev. A*, **17**, 455 (1978).
[15] F. Sagués and M. San Miguel, *Phys. Rev. A*, **32**, 1843 (1985).
[16] F. Haake, J. W. Haus and R. Glauber, *Phys. Rev. A*, **23**, 3255 (1981).
[17] A. J. F. Siegert, *Phys. Rev.*, **81**, 617 (1951); D. A. Darling and A. J. F. Siegert, *Ann. Math. Stat.*, **24**, 624 (1953).
[18] H. Risken, *The Fokker-Planck Equation* (Springer-Verlag, Berlin, 1984).
[19] M. Abramowitz and I. Stegun, *Handbook of Mathematical Functions* (Dover, New York, 1970).